Quadratic Functions and Their Properties

Quadratic Function: A function that can be written in the form $f(x) = ax^2 + bx + c$, where a, b, and c are real numbers and where $a \neq 0$. The domain of a quadratic function is the set of all real numbers.

Graphing a Quadratic Function Using Transformations

1. Begin with the parent function $f(x) = x^2$.
2. Rewrite the function in vertex form $f(x) = a(x-h)^2 + k$ by completing the square.
3. Transform with the following:

   a: If a is positive, the graph opens up. The y-coordinate of the vertex is a minimum value.
   If a is negative, the graph opens down. The y-coordinate of the vertex is a maximum value.
   If $|a| > 1$, the graph is narrower than the graph of $f(x) = x^2$.
   If $|a| < 1$, the graph is wider than the graph of $f(x) = x^2$.

   h: h controls the horizontal shift (left and right).

   k: k controls the vertical shift (up and down).

Vertex: (h, k) Axis of Symmetry: $x = h$

Completing the Square: Figuring out what constant to add to a binomial of the form $x^2 + bx$ to make it into a perfect square trinomial, then writing the result in factored form.

Completing the Square for the Binomial $x^2 + bx$

1. Divide the coefficient of the x-term by 2. $\left(\text{Find } \frac{b}{2}\right)$.
2. Square the answer from step 1. $\left(\text{Find } \left(\frac{b}{2}\right)^2\right)$.
3. Add the result of step 2 to the binomial.
4. Rewrite as a perfect square: $\left(x + \frac{b}{2}\right)^2$.

Example: Add the proper constant to the binomial to make it into a perfect square trinomial. Then factor the trinomial.

$x^2 + 12x + \_\_\_$	$x^2 - 7x + \_\_\_$
$\frac{12}{2} = 6$	$-7 \div 2 = -\frac{7}{2}$
$6^2 = 36$	$\left(-\frac{7}{2}\right)^2 = \frac{49}{4}$
$x^2 + 12x + 36$	$x^2 - 7x + \frac{49}{4}$
$(x+6)^2$	$\left(x - \frac{7}{2}\right)^2$

Examples: Add the proper constant to each binomial to make it into a perfect square trinomial. Then factor the trinomial.

a) $x^2 + 16x + 64 = (x + 8)^2$

$\frac{16}{2} = 8$ $8^2 = 64$

d) $x^2 - 3x + \frac{9}{4} = \left(x - \frac{3}{2}\right)^2$

$-\frac{3}{2}$ $\left(-\frac{3}{2}\right)^2 = \frac{9}{4}$

e) $x^2 + \frac{4}{3}x + \frac{4}{9} = \left(x + \frac{2}{3}\right)^2$

$\frac{4}{3} \div 2 = \frac{2}{3}$ $\left(\frac{2}{3}\right)^2 = \frac{4}{9}$

Writing $f(x) = ax^2 + bx + c$ in Vertex Form

1. Group ax^2 and bx together in parentheses.
2. If $a \neq 1$, factor out a from $ax^2 + bx$. Include a negative if the quadratic term is negative.
3. Complete the square (divide b by 2 and square the result). Add the answer inside the parentheses. Keep the equation balanced by adding or subtracting outside the parentheses. (You are adding 0 to one side of the equation.)
4. Write the expression inside the parentheses as a perfect square.

Examples: Write each equation in vertex form. Then find the vertex.

a) $f(x) = x^2 - 8x - 5$ $\quad \frac{-8}{2} = -4 \quad (-4)^2 = 16$

$f(x) = (x^2 - 8x \quad) - 5$

$f(x) = (x^2 - 8x + 16) - 5 - 16$

added zero

$\boxed{f(x) = (x-4)^2 - 21}$

vertex: $(4, -21)$

b) $f(x) = 3x^2 + 6x + 1$ $\quad \frac{2}{2} = 1 \quad 1^2 = 1$

$f(x) = (3x^2 + 6x) + 1$

$f(x) = 3(x^2 + 2x \quad) + 1$

$f(x) = 3(x^2 + 2x + 1) + 1 - 3$

$+3 \leftarrow$ added zero

$\boxed{f(x) = 3(x+1)^2 - 2}$

Vertex: $(-1, -2)$

c) $y = -x^2 + 4x - 3$ $\quad \frac{-4}{2} = -2 \quad (-2)^2 = 4$

$y = (-x^2 + 4x) - 3$

$y = -(x^2 - 4x \quad) - 3$

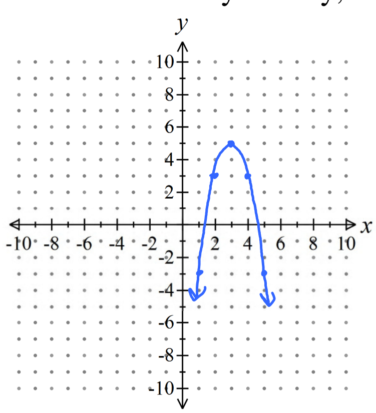
$y = -(x^2 - 4x + 4) - 3 + 4$

$-4 \leftarrow$ added zero

$\boxed{y = -(x-2)^2 + 1}$

vertex: $(2, 1)$

Example: Rewrite the function $f(x) = -2x^2 + 12x - 13$ in vertex form by completing the square. Find the vertex and axis of symmetry, then draw the graph. Find the maximum or minimum value.



$f(x) = (-2x^2 + 12x) - 13$

$f(x) = -2(x^2 - 6x \quad) - 13$

$f(x) = -2(x^2 - 6x + 9) - 13 + 18$

$-18 \leftarrow$ added zero

$\boxed{f(x) = -2(x-3)^2 + 5}$

vertex: $(3, 5)$

axis of symmetry: $x = 3$

maximum value $= 5$

The Vertex Formula

By completing the square, we can rewrite $f(x) = ax^2 + bx + c$ as $f(x) = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$.

This gives us a quick way to find the vertex when the equation is in standard form:

- The x-coordinate of the vertex is $-\frac{b}{2a}$.
- To find the y-coordinate, plug the x-coordinate into the original equation.

Properties of the Graph of a Quadratic Function

For the function, $f(x) = ax^2 + bx + c$, $a \neq 0$:

Vertex: $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$ **Axis of Symmetry:** The line $x = -\frac{b}{2a}$

Parabola opens up if $a > 0$; the vertex is a minimum point.
Parabola opens down if $a < 0$; the vertex is a maximum point.

Examples: Use the vertex formula to locate the vertex and axis of symmetry of the parabola. Determine whether the quadratic function has a maximum or minimum value, then find the value.

a) $f(x)=x^2-6x-8$

$x=\frac{6}{2(1)}=3$

$f(3)=3^2-6(3)-8=9-18-8=-17$

vertex $=(3,-17)$
axis of symmetry: $x=3$
minimum value $=-17$

b) $f(x)=-4x^2+2x+1$

$x=\frac{-2}{2(-4)}=\frac{1}{4}$

$f\left(\frac{1}{4}\right)=-4\left(\frac{1}{4}\right)^2+2\left(\frac{1}{4}\right)+1$

$=-4\left(\frac{1}{16}\right)+2\left(\frac{1}{4}\right)+1$

$=-\frac{1}{4}+\frac{1}{2}+1=\frac{5}{4}$

vertex: $\left(\frac{1}{4},\frac{5}{4}\right)$
axis of symmetry: $x=\frac{1}{4}$
maximum value $=\frac{5}{4}$

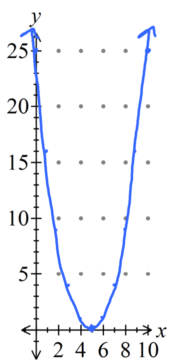
Graphing a Quadratic Function Using Its Vertex, Axis, and Intercepts

The y-intercept is the value of f at $x=0$; that is, the y-intercept is $f(0)=c$.

The x-intercepts, if any, are found by solving the quadratic equation $ax^2+bx+c=0$.

Discriminant	**Graph of $f(x)=ax^2+bx+c$**	**Solutions of $ax^2+bx+c=0$**
$b^2-4ac>0$	Two x-intercepts	Two real solutions
$b^2-4ac=0$	One x-intercept	One real solution
$b^2-4ac<0$	No x-intercepts	No real solutions

Example: Graph the function $f(x)=x^2-10x+25$ by determining whether the graph opens up or down and by finding its vertex, axis of symmetry, y-intercept, and x-intercepts, if any. Determine the domain and range of f and determine where f is increasing and where it is decreasing.



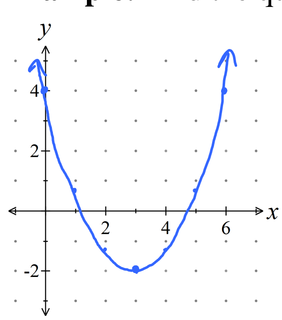
$f(x)=(x-5)^2$
opens up
vertex: $(5,0)$
axis of symmetry: $x=5$
y-intercept:
$f(0)=(0-5)^2=25$
$(0,25)$
x-intercept(s):
$(x-5)^2=0$
$x=5$
$(5,0)$

Domain: $(-\infty,\infty)$ or $\mathbb{R}$
Range: $[0,\infty)$ or $\{y \mid y\geq 0\}$
Increasing: $(5,\infty)$ or $\{x \mid x>5\}$
Decreasing: $(-\infty,5)$ or $\{x \mid x<5\}$

Writing a Quadratic Equation when You Know the Vertex and Another Point

1. Use vertex form: $y=a(x-h)^2+k$
2. Plug in the vertex for h and k.
3. Plug in the other point for x and y (or $f(x)$).
4. Simplify and solve for a. (Don't forget order of operations.)
5. Write your final answer by plugging a, h, and k back into vertex form.

Example: Find the quadratic function whose vertex is $(3,-2)$ and whose y-intercept is 4. Graph the function.



$y=a(x-h)^2+k$
Plug in vertex for (h,k):
$y=a(x-3)^2-2$
Plug in $(0,4)$ for (x,y):
$4=a(0-3)^2-2$
$4=9a-2$
$6=9a$
$a=\frac{2}{3}$

$y=\frac{2}{3}(x-3)^2-2$

	Standard Form	Vertex Form	Factored Form
Equation	$y = ax^2 + bx + c$	$y = a(x-h)^2 + k$	$y = a(x-p)(x-q)$
Vertex	Complete the square and write in vertex form. -or- $x = \frac{-b}{2a}$ Plug the x-coordinate into the equation to get the y-coordinate.	(h,k)	Find average of p and q. $x = \frac{p+q}{2}$ (The x-coordinate of the vertex is at the midpoint of the x-intercepts.) Plug the x-coordinate into the equation to get the y-coordinate.
***y*-intercept**	c (Replace x with zero. Solve for y.)	Replace x with zero. Solve for y.	Replace x with zero. Solve for y.
***x*-intercepts (roots, zeros, solutions)**	Replace y with zero. Solve for x by factoring or quadratic formula.	Replace y with zero. Solve for x by isolating the perfect square and using the square root principle. (Don't forget the ±.)	p and q (Replace y with zero. Solve for x using the zero product property.)

For all forms:

Direction of Opening	Up if a is positive Down if a is negative
Vertical Stretch	a
Counting Pattern (Shortcut)	Start at the vertex. Find more points by counting: $\leftrightarrow 1$, $\uparrow a$ $\leftrightarrow 1$, $\uparrow 3a$ $\leftrightarrow 1$, $\uparrow 5a$ $\leftrightarrow 1$, $\uparrow 7a$, etc. (If a is negative, move down instead of up.)